CSCI 340: Computational Models

Nonregular Languages

Nonregular Languages

Definition

A language that cannot be defined by a regular expression is called a **nonregular** language.

By Kleene's Theorem, a nonregular language can also not be accepted by any Finite Automaton (DFA or NFA) or by any Transition Graph.

Example

$$L = \{\lambda \;\; ab \;\; aabb \;\; aaabbb \;\; aaaabbbb \;\; \ldots\}$$

or alternatively defined as:

$$L = \{a^n b^n\}$$

The Pumping Lemma

Lemma

Let L be any regular language that has infinitely many words. Then there exists some three strings x, y, and z (where y is **not** the null string) such that all strings of the form

$$xy^n z \text{ for } n = 1\ 2\ 3\ \ldots$$

are words in L.

The Pumping Lemma

Proof (start...)

If L is a regular language, then there is an FA that accepts exactly the words in L and no more. This FA will have a finite number of states but infinitely many words. This means there is some cycle.

Let w be some word in L that has more letters in it than there are states in the machine. When this word generates a path through the machine, we **must** revisit a state that it has been to before.

Continuing the Proof of the Pumping Lemma (2/3)

Let us break up the word w into three parts:

1. Let x be all the letters of w starting at the beginning that lead up to the first state that is revisited. x may be the null string.
2. Let y denote the substring of w that travels around the "circuit" which loops. y cannot be the null string.
3. Let z be the rest of the letters in w that starts after y. This z could be null. The path for z could also possibly loop around the y-circuit (it's arbitrary).

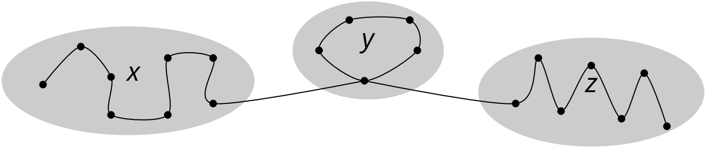
Clearly, from this definition given above,

$$w = xyz$$

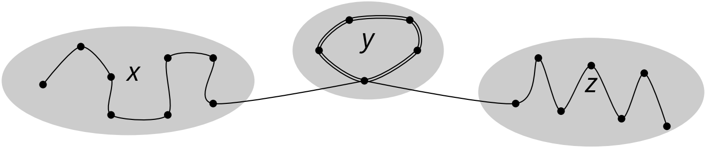
and w is accepted by this machine.

Continuing the Proof of the Pumping Lemma (3/3)

Q1: What is the path through this machine of the input string *xyz*?

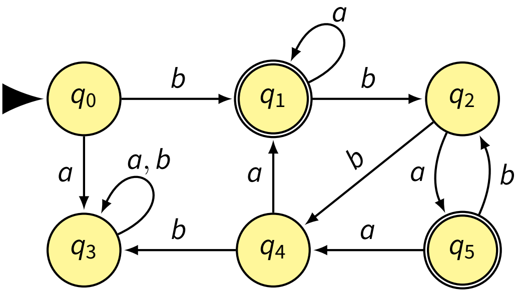


Q2: What is the path through this machine of the input string *xyyz*?



Note: All languages **L** *must be of the form* $w = xy^n z$ *for this to be "accepted". If they were not of this form, then the FA would not have such a trace.*

Example



$$w = bbbababa$$

$$\begin{array}{cccc} w = & b & baa & baba \\ & x & y & z \end{array}$$

What would happen when $w = xyyz = b \;\; bba \;\; bba \;\; baba$?

Show L is Non-regular with the Pumping Lemma

Suppose for a moment that we never talked about $L = \{a^n b^n\}$

The pumping lemma states there must be strings x,y, and z such that all words of the form $xy^n z$ are in L. Is this possible?

$$aaa \ldots aaaabbbb \ldots bbb$$

- If y is made entirely of a's then when we pump to $xyyz$, the word will have more a's than b's.
- If y is made entirely of b's then when we pump to $xyyz$, the word will have more b's than a's.
- y **must** be made up of some number of a's followed by some number of b's. This means $xyyz$ would have two copies of the substring ab. Our original language prohibits this. Therefore, $xyyz$ cannot be a word in L. And L is not regular.

Another Example of Showing L is Non-regular

Once we have shown $\{a^n b^n\}$ is non-regular, we can show that the language EQUAL (all words with the same total number of a's and b's) is also non-regular.

- The language $\{a^n b^n\}$ is the *intersection* of all words defined by the regular expression $\mathbf{a}^*\mathbf{b}^*$ and the language EQUAL.

$$\{a^n b^n\} = \mathbf{a}^*\mathbf{b}^* \cap \text{EQUAL}$$

- If EQUAL were a regular language, then $\{a^n b^n\}$ would be the intersection of two regular language (as discussed in Chapter 9). Additionally, it would need to be regular itself (which it is not).
- Therefore, EQUAL cannot be regular since $\{a^n b^n\}$ is non-regular.

Yet Another Non-regular Language

Consider the language $L = a^n b a^n = \{b \;\; aba \;\; aabaa \;\; aaabaaa \;\; \ldots\}$. If this language were regular, then we know the Pumping Lemma would have to hold true.

- xyz and $xyyz$ would both need to be in L
- *Observation 1:* If the y string contained the b, then $xyyz$ would contain two b's. This is not possible – $xyyz$ is not part of L
- *Observation 2:* If the y string contained all a's then the b in the middle is either on the x or z side. In either case, $xyyz$ would increase the number of a's either before or after the b
- *Conclusion 1:* $xyyz$ does not have b in the middle and is not of the form $a^n b a^n$
- *Conclusion 2:* L cannot be pumped and is therefore not regular

Additional Examples (on Chalkboard)

1. $a^n b^n a b^{n+1}$
2. PALINDROME
3. PRIME $= \{a^n$ where p is a prime$\}$

Plus a Stronger Theorem

Let L be an infinite language accepted by a finite automaton with N states. Then for all words w in L that have more than N letters, there are strings x, y, and z, where y is not null and length(x) + length(y) does not exceed N such that

$$w = xyz$$

and all strings of the form

$$xy^n z(\text{ for } n = 1\ 2\ 3\ \ldots)$$

are in L $\square$

Limitations of the pumping lemma

The pumping lemma is *negative* in its application. It can only be used to show that certain languages are not regular.

- Let's consider some FA – each state (final or non-final) can be thought of as creating a society of a certain class of strings.
- If there exists a string formed by some path leading to a state, it is part of that state's society.
- If string x and string y are in the same society, then for all other strings z, either xz and yz are both accepted or rejected

Theorem (The Myhill-Nerode Theorem)

Given a language L, we shall say two string x and y are in the same class if for all possible strings z, xz and yz are both in L or both are not

1. *The language L divides the set of all strings into separate classes*
2. *If L is regular, the number of classes L creates is finite.*
3. *If the number of classes L creates is finite, then L is regular*

Proving the Myhill-Nerode Theorem

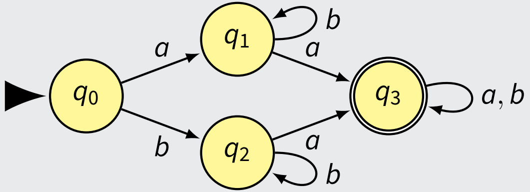
Proof by contradiction – Part 1.

- Split classes in an intentionally bad way: Suppose any two students at college are in the same class if the have taken a course together
- *A* and *B* may have taken history together, *B* and *C* may have taken geography together, but *A* and *C* never took a class together. Then *A*, *B*, and *C* are not all in the same class.
- If *AZ* and *BZ* are always in *L* and *BZ* (or not) and *CZ* are always in *L* (or not), then *A*, *B*, and *C* must all be in the same class
- If *S* is in a class with *X* and *S* is also in a class with *Y*, then by reasoning above *X* and *Y* must be in the same class.
- Therefore, *S* cannot be in two different classes. No string is in two different classes and every string **must** be in some class.
- Therefore, every string is in exactly one class □

Proving the Myhill-Nerode Theorem

Proof of Part 2.

- If L is regular, then there is some FA that accepts L.
- Its finite number of states create a finite division of all strings into a finite number of societies.
- The problem is that two different states may define societies that are actually the same class



- Society "class" of q_1 and q_2: any word in them when followed by a string z will be accepted IFF z contains an a
- Since the societies are in the same class, and there are finitely many societies, there must be a finite number of classes. □

Proving the Myhill-Nerode Theorem

Proof by (pseudo-)construction – Part 3.

Let the finitely many classes be $C_1, C_2, \ldots C_n$ where C_1 is the class containing λ. We will transform these classes into an FA by showing how to draw the edges between (and assign start and final states)

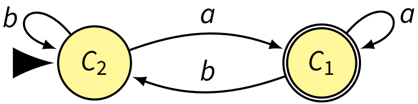
1. The start state must be C_1 because of λ
2. If a class contains one word of L then $w \in L \ \forall w \in C$. Let $s \in L, w \in L \mid w \in C_k$. When $z = \lambda, w\lambda \in L \wedge s\lambda \in L$ (or not). Label all states that are subsets of L as final states.
3. Repeat the following for all classes C_m:
   If $x \in C_m \wedge y \in C_m$, then $\forall z \ (xz \in L \wedge yz \in L)$.
   Let $C_a = xa \, \forall x \in C_m$. Draw an a-edge from C_m to C_a.
   Let $C_b = xb \, \forall x \in C_m$. Draw an b-edge from C_m to C_b.
4. Once outgoing edges are drawn for all classes, we have an FA

$\square$

Examples using Myhill-Nerode Theorem

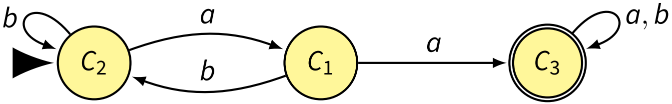
All words that end in *a*

- C_1 – all strings that end in a (final)
- C_2 – all strings that don't end in a (start)



All words that contain a double *a*

- C_1 – strings without aa that end in a
- C_2 – strings without aa that end in b or λ
- C_3 – strings with aa



Examples using Myhill-Nerode Theorem

Showing languages are regular

- EVEN-EVEN
- two or more *b*'s
- start and end with the same letter

Showing languages are non-regular

- $a^n b^n$
- $a^n b a^n$
- EQUAL
- PALINDROME

Examples using Myhill-Nerode Theorem

Showing languages are regular

- EVEN-EVEN
- two or more *b*'s
- start and end with the same letter

Showing languages are non-regular

- $a^n b^n$ We only need to observe that $a, aa, aaa, \ldots$ are all in different classes because there's exactly b^m that will match a^m
- $a^n ba^n$
- EQUAL
- PALINDROME

Examples using Myhill-Nerode Theorem

Showing languages are regular

- EVEN-EVEN
- two or more b's
- start and end with the same letter

Showing languages are non-regular

- $a^n b^n$ We only need to observe that $a, aa, aaa, \ldots$ are all in different classes because there's exactly b^m that will match a^m
- $a^n ba^n$ The strings $ab, aab, aaab, \ldots$ are all in different classes because we need a matching ba^m for each class
- EQUAL
- PALINDROME

Examples using Myhill-Nerode Theorem

Showing languages are regular

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Showing languages are non-regular

- $a^n b^n$ We only need to observe that $a, aa, aaa, \ldots$ are all in different classes because there's exactly b^m that will match a^m
- $a^n ba^n$ The strings $ab, aab, aaab, \ldots$ are all in different classes because we need a matching ba^m for each class
- EQUAL Because for each of the strings $a, aa, aaa, aaaa, \ldots$ some $z = b^m$ will be alone in EQUAL
- PALINDROME

Examples using Myhill-Nerode Theorem

Showing languages are regular

- EVEN-EVEN
- two or more *b*'s
- start and end with the same letter

Showing languages are non-regular

- $a^n b^n$ We only need to observe that $a, aa, aaa, \ldots$ are all in different classes because there's exactly b^m that will match a^m
- $a^n ba^n$ The strings $ab, aab, aaab, \ldots$ are all in different classes because we need a matching ba^m for each class
- EQUAL Because for each of the strings $a, aa, aaa, aaaa, \ldots$ some $z = b^m$ will be alone in EQUAL
- PALINDROME $ab, aab, aaab, \ldots$ are all in different classes. For each, one value of $z = a^m$ will create a PALINDROME when added but to no other

Bonus: Prefixes

Definition

If R and Q are languages, then the language "the prefixes of Q in R," denoted by the symbolism **Pref(**Q **in** R**)** is the set of all strings of letters that can be concatenated to the front of some word in Q to produce some word in R

$$\text{Pref}\,(Q \text{ in } R) = \text{all strings } p \text{ such that } q \in Q, w \in R \mid pq = w$$

Theorem

If R is regular and Q is **any** *language whatsoever, then the language*

$$P = \text{Pref}(Q \text{ in } R)$$

is regular

Homework 6b

1. Use the pumping lemma, show each are non-regular
   1. $a^n b^{n+1}$
   2. $a^n b^n a^n$
2. Using Myhill-Nerode theorem, show each are non-regular
   1. EVEN-PALINDROME (all PALINDROMEs with even length)
   2. SQUARE ($a^{n^2} \mid n \geq 1$)
3. Let us define PARENTHESES to be the set of all algebraic expressions where everything but parentheses have been deleted e.g. $\{\lambda\ ()\ (())\ ()()\ ((()))\ (())()\ ()(())\ ()()()\ \ \ldots\}$
   1. Show its non-regular using Myhill-Nerode
   2. Show the pumping lemma can't prove that it's non-regular
   3. If we convert (to a and) to b, show that PARENTHESES becomes a subset of EQUAL in which each word has the property that when read from left-to-right, there are never more b's than a's