CSCI 340: Computational Models

Languages

What is a Language?

- English: “letters”, “words”, “sentences”
- Programming: “keywords”, “variables”, “numbers”, “symbols”
- General: *language structure* – decision of whether a given string of units is “matched” or *valid*

Important Terms

- *alphabet* – finite set of fundamental units out of which we build structures.
- *language* – a certain specified set of strings of characters from the alphabet
- *words* – strings which are permissible in the language
- *empty string* or *null string* – a string which has no letters (λ)
- *null set* – denoted as $\varnothing$

Question

Is there a difference between empty string and an empty language?

An Aside on Set Theory

Assume

- L is a language
- $+$ is "union of sets" operator
- $\varnothing$ is empty set
- λ is empty string

Claim 1

$L + \{\lambda\} \neq L$

Claim 2

$L + \varnothing = L$

This implies that $\varnothing$ is a valid definition for a language

The English Languages

Alphabet

$\Sigma = \{a\ b\ c\ d\ e\ \ldots\ z\ '\ -\}$

Words

$\mathit{ENGLISH\text{-}WORDS} = \{\text{all the words in a standard dictionary}\}$

Problem: How can we represent sentences?

The *Real* English Languages

Alphabet

$\Gamma =$ entries of *ENGLISH-WORDS* $+ \{$*space*$\} + \{$*punctuation*$\}$

Words (a.k.a. English Sentences)

- Must rely on grammatical rules of English
- There are *infinitely many*
  - I ate one apple.
  - I ate two apples.
  - I ate three apples.
  - . . . .

We can list all rules of the grammar to give a *finite description* for an *infinite language*. This will make “I ate three Tuesdays” valid!

Defining a Language

Language Defining Rules

1. Tell us how to test a string of alphabet letters that we are presented with
2. Tell us how to construct all of the words in the language by some clear procedure

Example

$\Sigma = \{x\}$

$L_1 = \{x \; xx \; xxx \; xxxx \; \ldots\}$
alternatively,
$L_1 = \{x^n \text{ for } n = 1 \; 2 \; 3 \; \ldots\}$

Working with a Language

Null String?

A language does not need to accept λ. L_1 doesn't

Concatenation

- Two strings written side by side yield a new string
- x^n concatenated with x^m is x^{n+m}

Symbols

- We can designate a word in a given language by a new symbol
  - Let $a = xx$ and $b = xxx$
  - Therefore, $ab = xxxxxx$
- Two words of L concatenated are not guaranteed to produce another word in L

Example: Numbers

Example

$\Sigma = \{$0 1 2 3 4 5 6 7 8 9$\}$

$L_3 = \{$ any finite string of Σ letters that doesn't start with 0$\}$

A subset of L_3 might *look like*:

$L_3 = \{$1 2 3 4 5 6 7 8 9 10 11 12 $\ldots\}$

If we want to allow the string (word) 0, we could say:

$L_3 = \{$ any finite string of Σ letters that, if it starts with 0,
has no more letters after the first $\}$

Example: Length

We define the function **length** of a string to be the number of letters in the string. We write this function using the word “length”. For example, if $a = xxxx$ in the language L_1, then

$$\text{length}(a) = 4$$

Or we could write directly that in a language, such as L_3,

$$\text{length}(428) = 3$$

In any language which includes λ we have

$$\text{length}(\lambda) = 0$$

Corollary: For any word w in a language, if length(w) = 0, then $w = \lambda$

Redefining Number with **length**

We can present another definition for L_3

$$L_3 = \{ \text{any finite string of } \Sigma \text{ letters that, if it has length more than } 1, \text{ does not start with a } 0 \}$$

This isn't necessarily a better definition, but it illustrates equivalent languages can be defined in multiple ways.

Adding λ to a finite language

If we look back to L_1, which described one or more "x" characters defining valid words, we may want to expand the language to include *empty string*

$$L_4 = \{\lambda\ x\ xx\ xxx\ xxxx\ \ldots\}$$

Alternatively,

$$L_4 = \{x^n \text{ for } n = 0\ 1\ 2\ 3\ \ldots\}$$

Notice: $x^0 = \lambda$

Example: Reverse

Definition

Let us introduce the function **reverse**. If a is a word in some language, L, then reverse(a) is the same string of letters spelled backward even if this backwards string is not a word in L.

Example

$$\text{reverse}(xxx) = xxx$$
$$\text{reverse}(xxxxx) = xxxxx$$
$$\text{reverse}(145) = 541$$

But let us also note that in L_1,

$$\text{reverse}(140) = 041$$

which is not a word in L_1

Example: Palindrome Language

Definition

PALINDROME (P) is a new language over the alphabet

$$\Sigma = \{a\ b\}$$

$$P = \{\lambda, \text{and all strings } x \mid \text{reverse}(x) = x\}$$

$$\therefore$$

$$P = \{\lambda\ a\ b\ aa\ bb\ aaa\ aba\ bab\ bbb\ aaaa\ abba\ \ldots\}$$

Interesting Properties

1. *concatenating* two words from P sometimes produces a word within P. e.g. $abba + abba = abbaabba$
2. More often than not, *concatenating* two words from P does not yield a word within P. e.g. $aa + aba = aaaba$

Kleene Closure (or the Kleene Star)

Definition

- Given an alphabet Σ, we wish to define a language in which any string of letters from Σ is a word, even the null string λ.
- This language shall be known as the **closure** of the alphabet.
- Symbolically denoted as: Σ^*

Example

If $\Sigma = \{x\}$, then $\Sigma^* = \{\lambda\ x\ xx\ xxx\ xxxx\ \ldots\}$

Example

If $\Sigma = \{0\ 1\}$, then $\Sigma^* = \{\lambda\ 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ \ldots\}$

Example

If $\Sigma = \{a\ b\ c\}$, then $\Sigma^* = \{\lambda\ a\ b\ c\ aa\ ab\ ac\ ba\ bb\ bc\ ca\ cb\ cc\ aaa\ \ldots\}$

Kleene Closure

- an **operation** that makes an infinite language or strings of letters out of an alphabet
- infinitely many words, each of a finite length
- often ordered by *size* first, then *lexicographically*

Definition

If S is a set of words, then S^* means the set of all finite strings formed by **concatenating** words from S. Any word may be used as often as we like, and λ is also included.

Problem

Compare:

ENGLISH-WORDS* and ENGLISH-SENTENCES

Kleene Closure

Example

$$S = \{aa\ b\}$$
$$S^* = ?$$

Example

$$S = \{a\ ab\}$$
$$S^* = ?$$

To prove that a certain word is in the closure language S^*, we must show how it can be written as a **concatenation** of words from the base set S.

Factor

The **concatenation** of words from a base set S can be viewed as a *factor* of a word from *closure* set S^*

Example

$S = \{xx\ xxx\}$
$S^* = \{x^n \text{ for } n = 0\ 2\ 3\ 4\ \ldots\}$

Notice how the word x is the only word not in the language S^*

There is also ambiguity in factoring certain strings e.g. *xxxxxxx*

$$(xx)(xx)(xxx) \quad \text{or} \quad (xx)(xxx)(xx) \quad \text{or} \quad (xxx)(xx)(xx)$$

How can we **prove** that S only contains repetitions of letter x not equal to size of 1?

Proving S^* contains all $x^n \mid n \neq 1$

Example

$S = \{xx\ xxx\}$
$S^* = \{x^n \text{ for } n = 0\ 2\ 3\ 4\ \ldots\}$

Proof (by constructive algorithm).

Base: $x^0 = \lambda$
Base: $x^2 = xx$
Base: $x^3 = xxx$

Factor: $x^4 = x^2 + x^2$
Factor: $x^5 = x^3 + x^2$

$x^{n+2} = x^n + x^2$ $\square$

Kleene Closure

The Kleene closure of two sets can end up being the **same language**

Example

$S = \{a\ b\ ab\}$
$T = \{a\ b\ bb\}$

- Both S^* and T^* define languages of all strings of a's and b's.
- Any string of a's and b's can be factored into syllables (a) and (b)

Consider *ababbabba* and *abababbbb*

$^+$ Notation

If for some reason we wish to modify the concept of closure to refer to only the concatenation of some *non-zero* strings from a set S, we use the notation $^+$ instead of *

Example

$$\text{If } \Sigma = \{x\}, \quad \text{then } \Sigma^+ = \{x \; xx \; xxx \; \ldots\}$$

- This is often referred to as *positive closure* ("one-or-more")
- If S is a language which contains λ, then $S^+ = S^*$
- If S is a language which doesn't contain λ, then $S^+ = S^* - \{\lambda\}$

Double Closure

Given S^*, apply its closure: $(S^*)^*$

- If S is not $\varnothing$ or $\{\lambda\}$, then S^* is infinite
- We will be taking the *closure* of an infinite set
- Arbitrary concatenation of the alphabet, applied *twice*

Proving $S^* = S^{**}$ (by construction).

$S = \{a\ b\}$	
$s = aababaaaaaba$	[arbitrary string]
$s = (aaba)(baaa)(aaba)$	[constructed from S^*]
$s = [(a)(a)(b)(a)][(b)(a)(a)(a)][(a)(a)(b)(a)]$	[constructed from S^{**}]
$s = (a)(a)(b)(a)(b)(a)(a)(a)(a)(a)(b)(a)$	[converted from S^{**} to S^*]
$S^{**} \subset S^*$	$[\forall e \in S^{**}, e \in S^*]$
$S^* \subset S^{**}$	$[\forall e \in S^*, e \in S^{**}]$
$S^* = S^{**}$	$\square$

Homework 1a

1. Consider the language S^*, where $S = \{aa\ b\}$. How many words does this language have of length 4? of length 5? of length 6? What can be said in general?
2. Consider the language S^*, where $S = \{aa\ aba\ baa\}$. Show that the words *aabaa*, *baaabaaa*, and *baaaaababaaaa* are all in this language. Can any word in this language be interpreted as a string of elements from S in two different ways? Can any word in this language have an odd total number of a's?
3. Prove that for all sets S,
   1. $(S^+)^* = (S^*)^*$
   2. $(S^+)^+ = S^+$
   3. Is $(S^*)^+ = (S^+)^*$ for all sets S?