Divide and Conquer Algorithms

Drawing a Ruler

Problem: Draw marks at regular intervals on a line

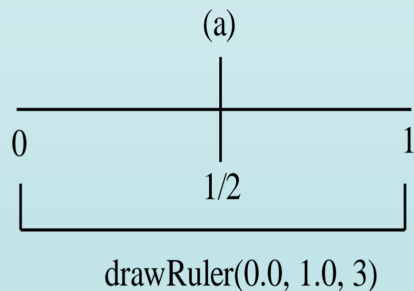
Write recursive

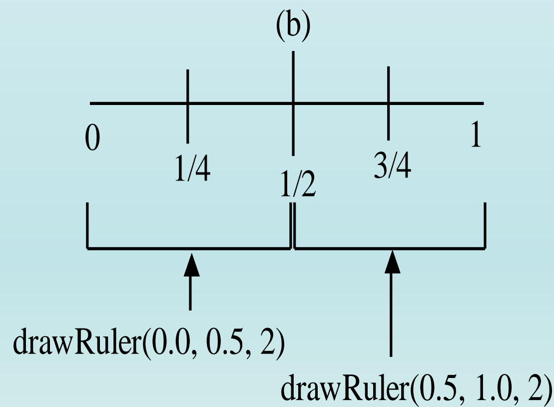
drawRuler (xBegin, xEnd, height)

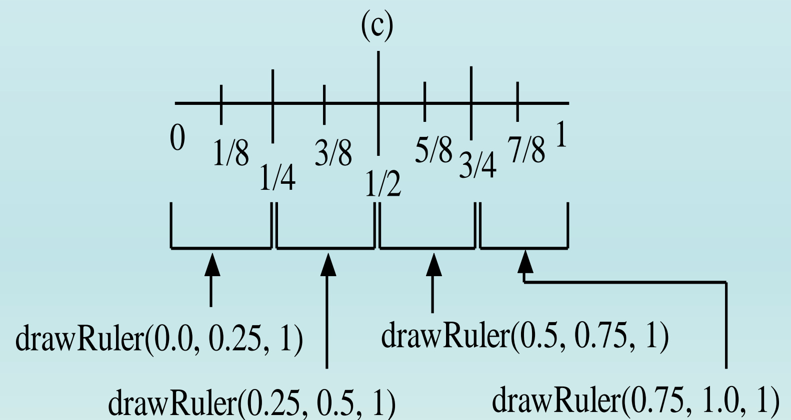
assuming

drawMark (xCoord, height) is written

and a height of 0 indicates no further divisions







Recursive Ruler

- drawRuler (begin, end, h)
  - Base case?
  - …

- A *Divide and Conquer (and Combine)* algorithm
  - Split problem into subproblems
  - Solve subproblems
  - Combine solutions to subproblems to solve whole problem

Iterative Ruler

```
void
drawRuler (begin, end, incr)
{
  // Start edge
  drawMark (begin, END_HEIGHT);
  begin += incr;
  while (begin != end)
  {
    drawMark (begin, ?);
    begin += incr;
  }
  // End edge
  drawMark (end, END_HEIGHT);
}
```

Two Sorting Algorithms

- Divide and Conquer Sorting Algorithms
  - Merge Sort
    - Split in half (divide)
    - Sort halves recursively (conquer)
    - Merge sorted halves (combine)
    - *Worst = Average = O(N lg N)*

  - Quick Sort
    - Choose pivot value from array
    - Place pivot in final position (divide)
      - Elements to left <= pivot
      - Elements to right >= pivot
    - Sort 2 sublists recursively (conquer)
    - Average = O(N lg N)
    - *Worst = $O(N^2)$*

Merge Sort

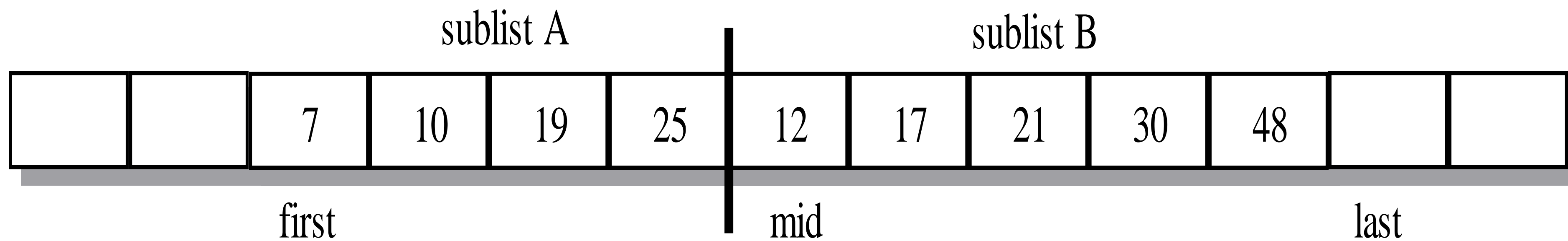
```
void
mergeSort (int A[], size_t first, size_t last)
{
  if (last – first > 1) {
   size_t mid = first + (last - first) / 2;
   mergeSort (A, first, mid);
   mergeSort (A, mid, last);
   inplace_merge (A + first, A + mid, A + last);
  }
}
```

Merge Algorithm

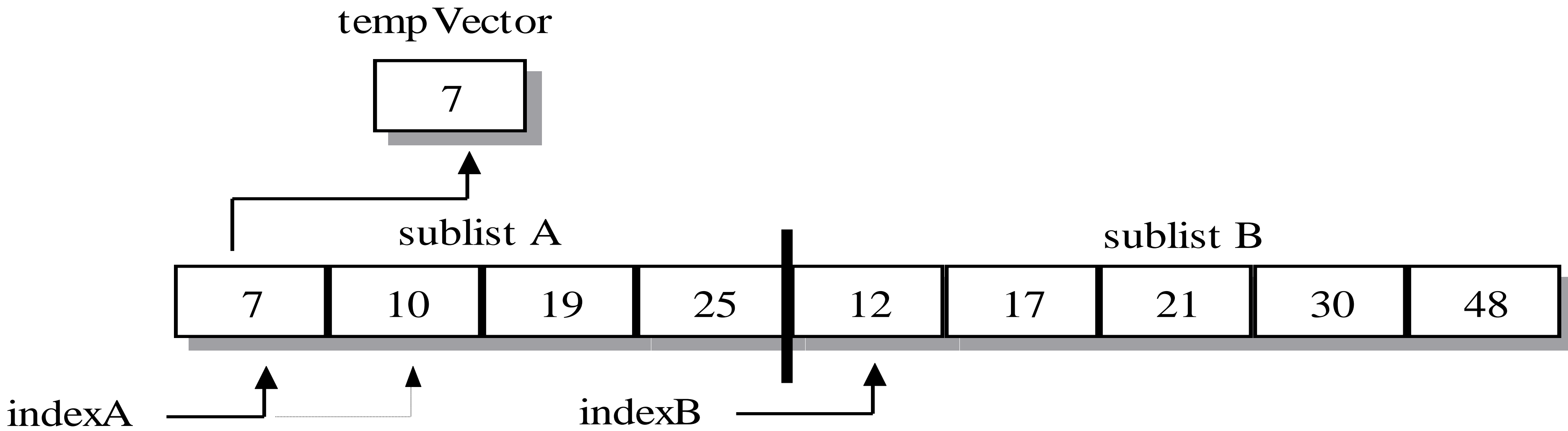
- Merge sort requires *merge* algorithm
  - Complexity of merge?

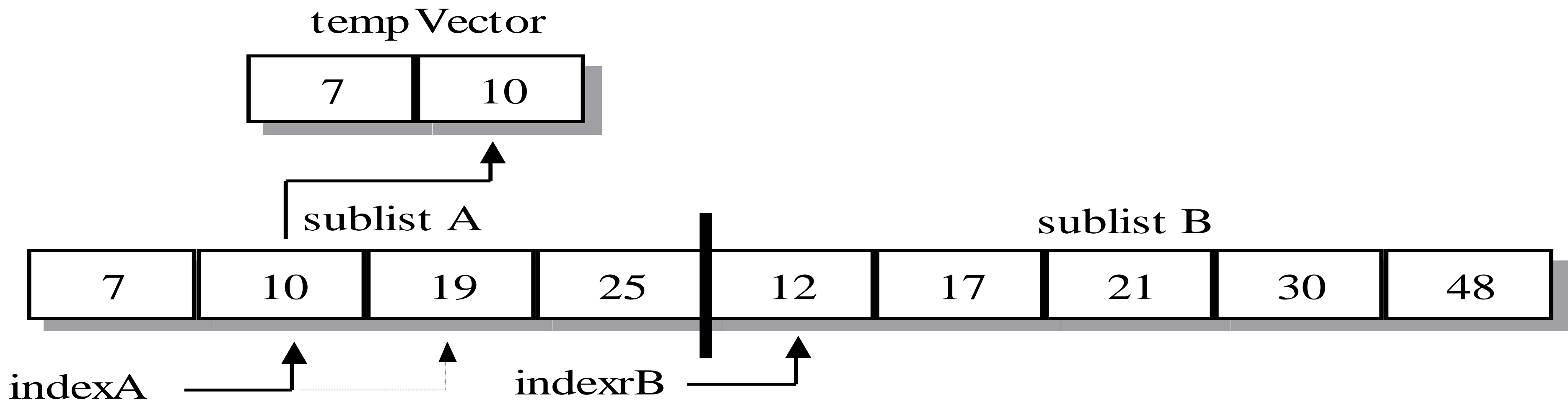
- Out-of-place merge: merge (v, first, mid, last)
  - v[first, mid) sorted
  - v[mid, last) sorted

  - Result: v[first, last) sorted

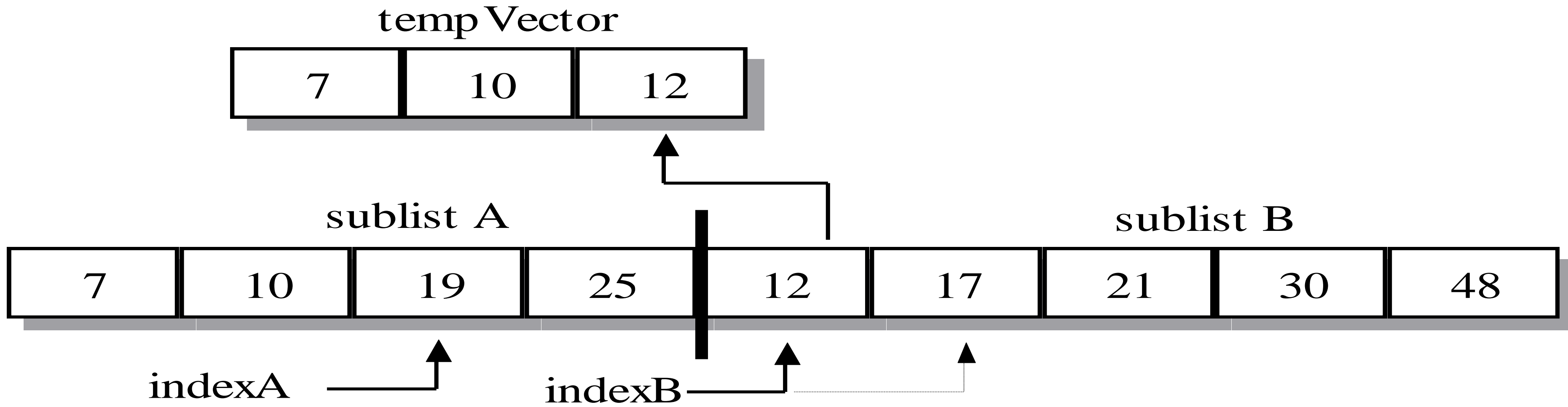


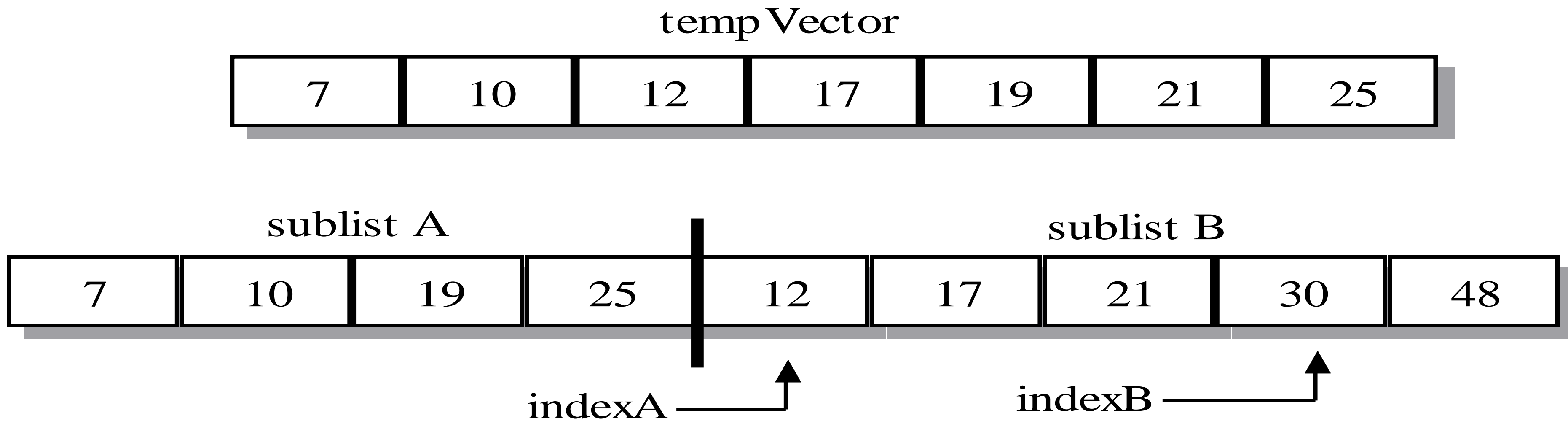
Merge Algorithm (Cont'd)



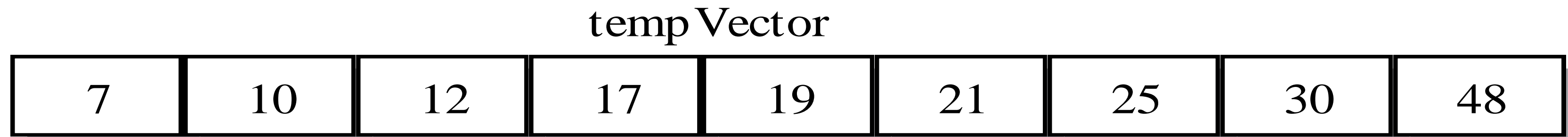


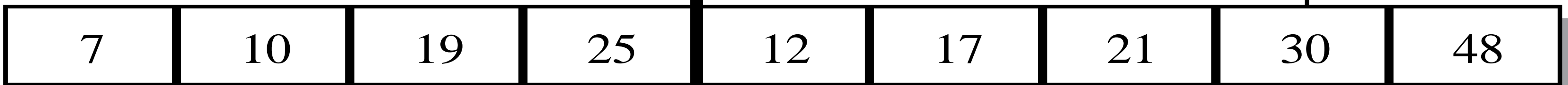
Merge Algorithm (Cont'd)

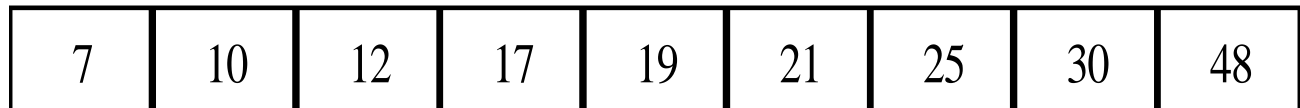




Merge Algorithm (Cont'd)

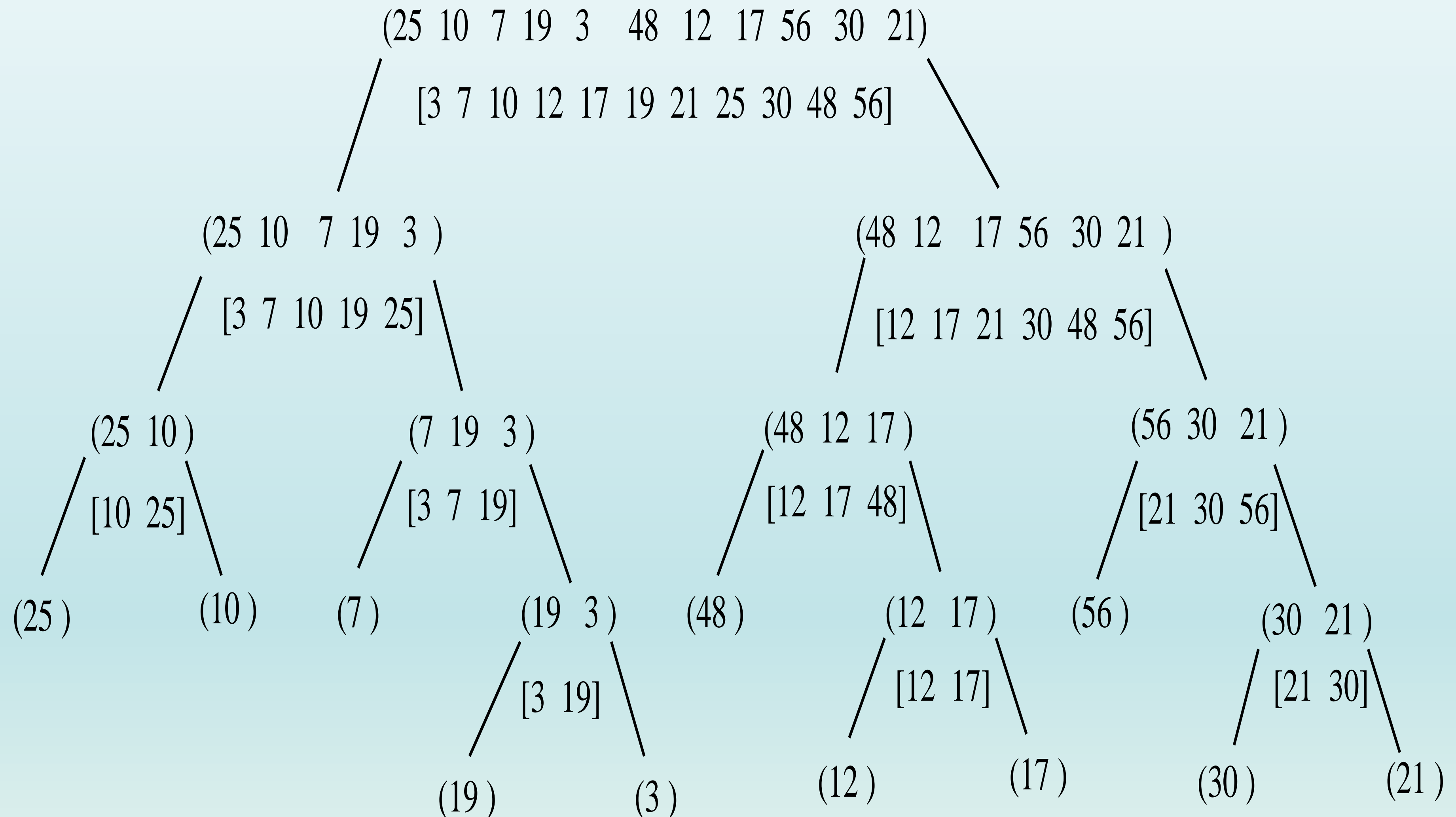




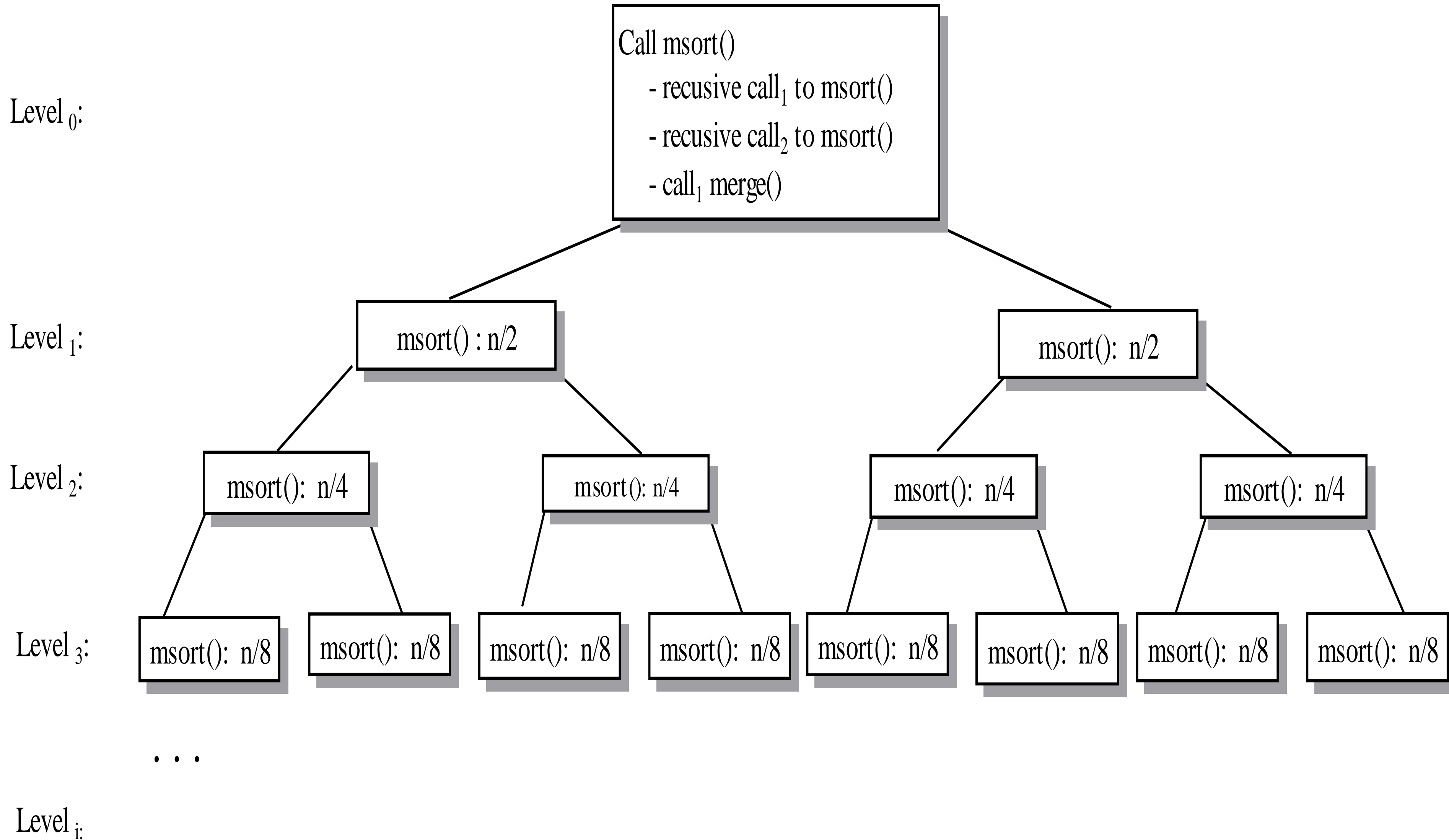


Partitioning and Merging of Sublists in Merge Sort

Postorder algorithm



Call Tree for Merge Sort



Quicksort Code

```
void
quickSort (int A[], size_t first, size_t last)
{
  if (last - first > 1)  {
    auto i = partition (A, first, last);
    // Pivot is placed in A[i]
    quickSort (A, first, i);
    quickSort (A, i + 1, last);
  }
}
```

Partition Routine

```
// Will go into details later...
size_t
partition (int A[], size_t first, size_t last) {
  median3 (A, first, last - 1);
  int pivot = A[last - 2];
  size_t up = first, down = last - 2;
  for ( ; ; ) {
      while (A[++up] < pivot)  { }
      while (A[--down] > pivot)  { }
      if (up >= down) break;
      swap (A[up], A[down]);
  }

  swap (A[last - 2], A[up]);
  return up;
}
```

Quicksort

- *Fastest* known sorting algorithm in practice
  - Part of std::sort
  - May be used in cstdlib qsort
    - qsort (void* base, size_t num, size_t size,
      int (*comp)(const void*, const void*))
- Average case: O(N log N)
- *Worst case*: $O(N^2)$

Quick Sort

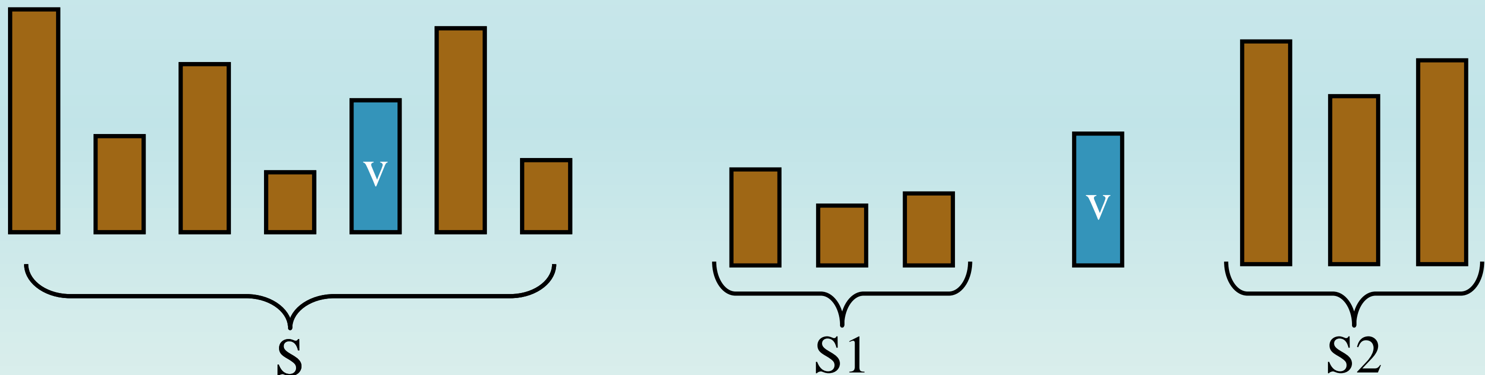
- Divide step
  - Pick any element (*pivot*) **v** in S
  - Partition S – {v} into two disjoint groups

    $S1 = \{x \in S - \{ v \} \mid x \leq v\}$

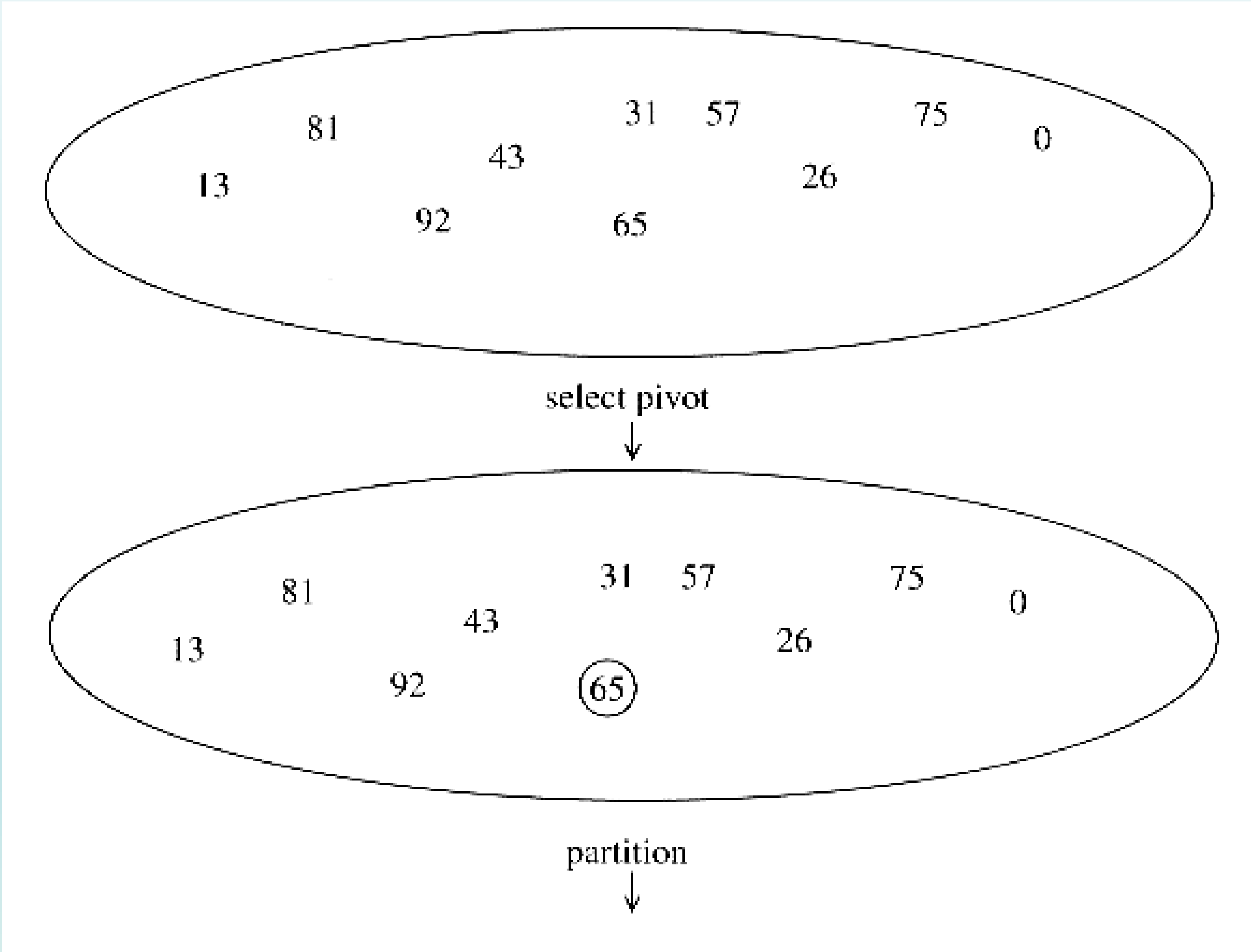
    $S2 = \{x \in S - \{ v \} \mid x \geq v\}$

- Conquer step: recursively sort S1 and S2

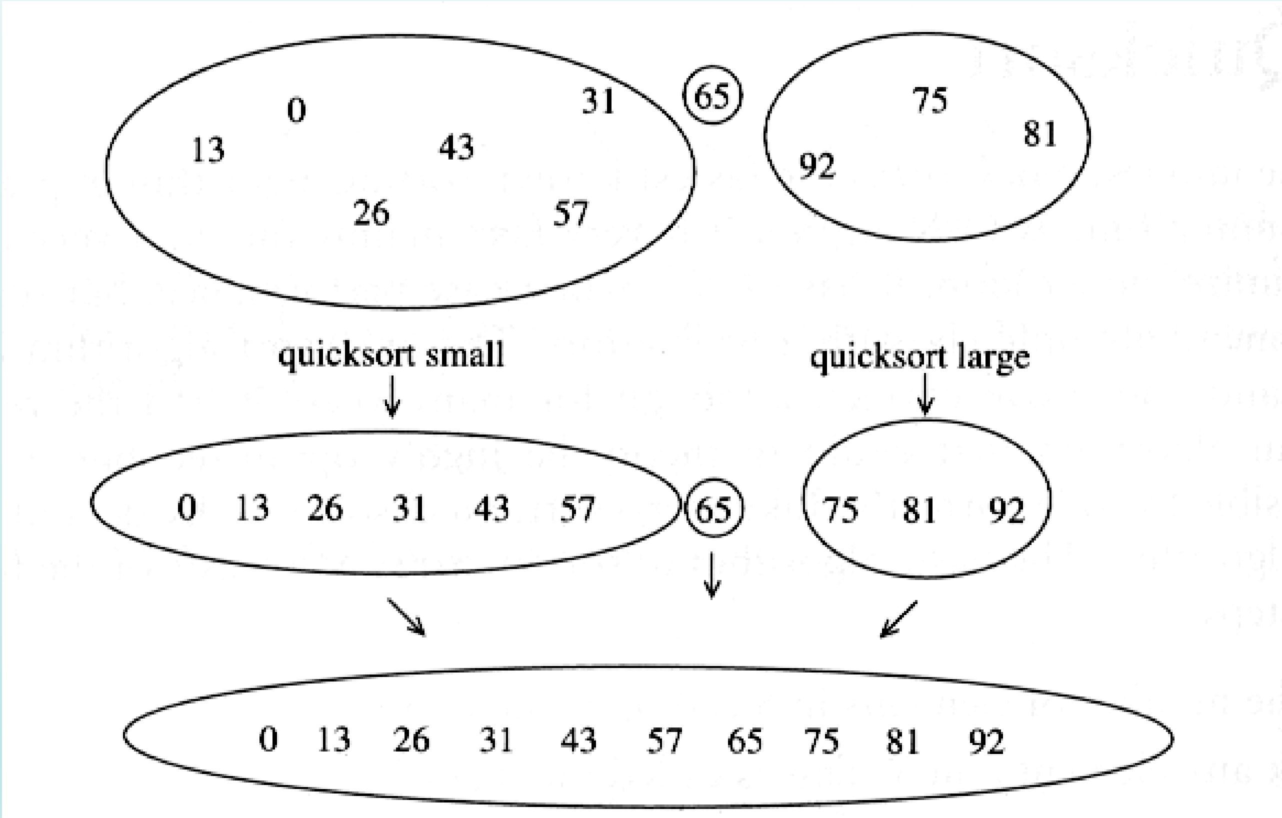
- Combine step: list the sorted S1, followed by v, followed by sorted S2



Quick Sort...



Quick Sort...

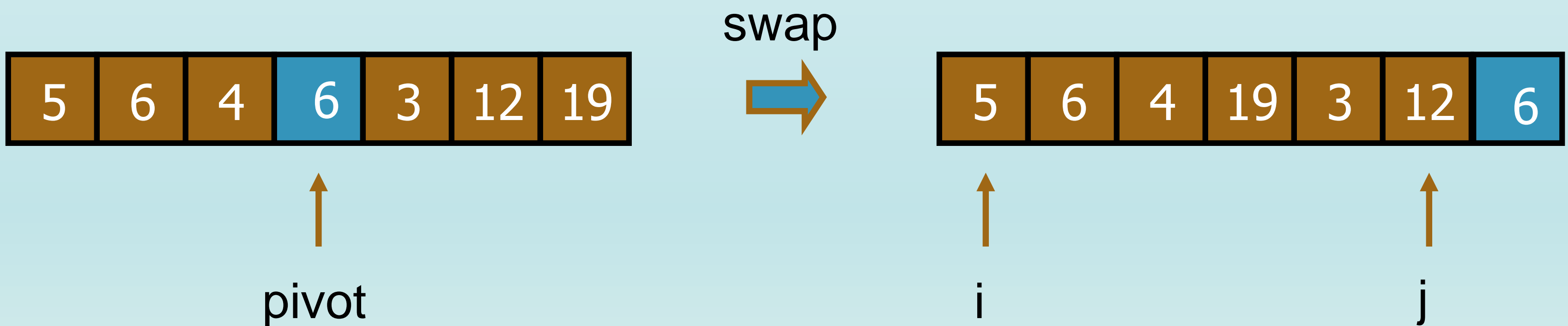


Partitioning

- Partitioning
  - Key step of quicksort algorithm
  - Many ways to implement
- How to pick pivot will be discussed later

Partitioning Strategy

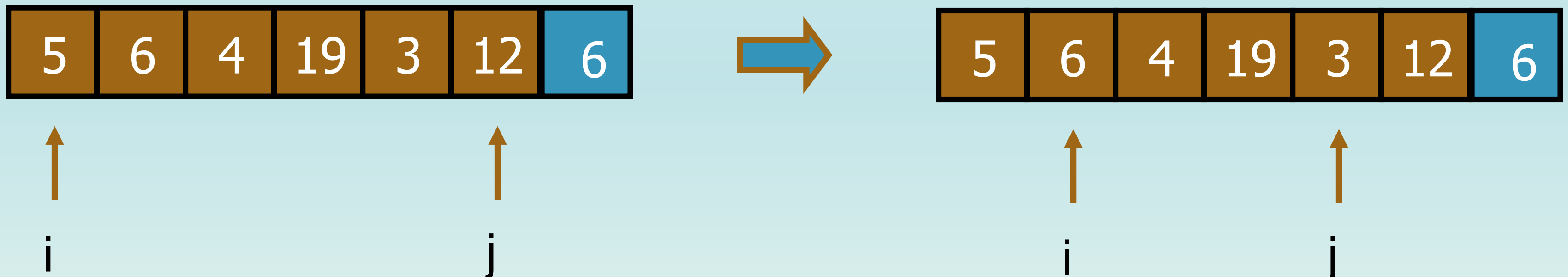
- Want to partition an array A[left .. right]
- Swap pivot and A[right]
- Let i = left; j = right - 1



Partitioning Strategy

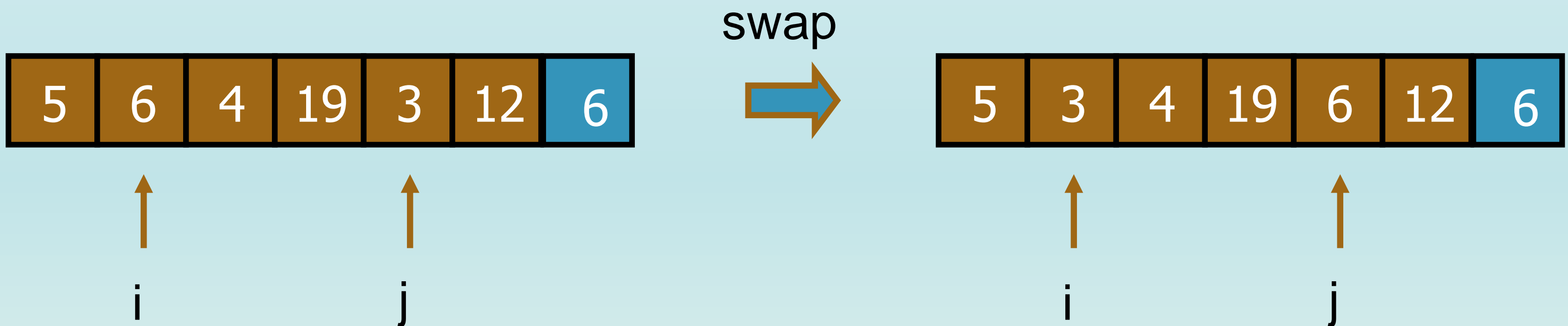
- Want to have
  - A[k] <= pivot, for k < i
  - A[k] >= pivot, for k > j

- While i < j
  - Move i right, skipping over elements smaller than pivot
  - Move j left, skipping over elements greater than pivot
  - When both i and j have stopped
    - A[i] >= pivot
    - A[j] <= pivot



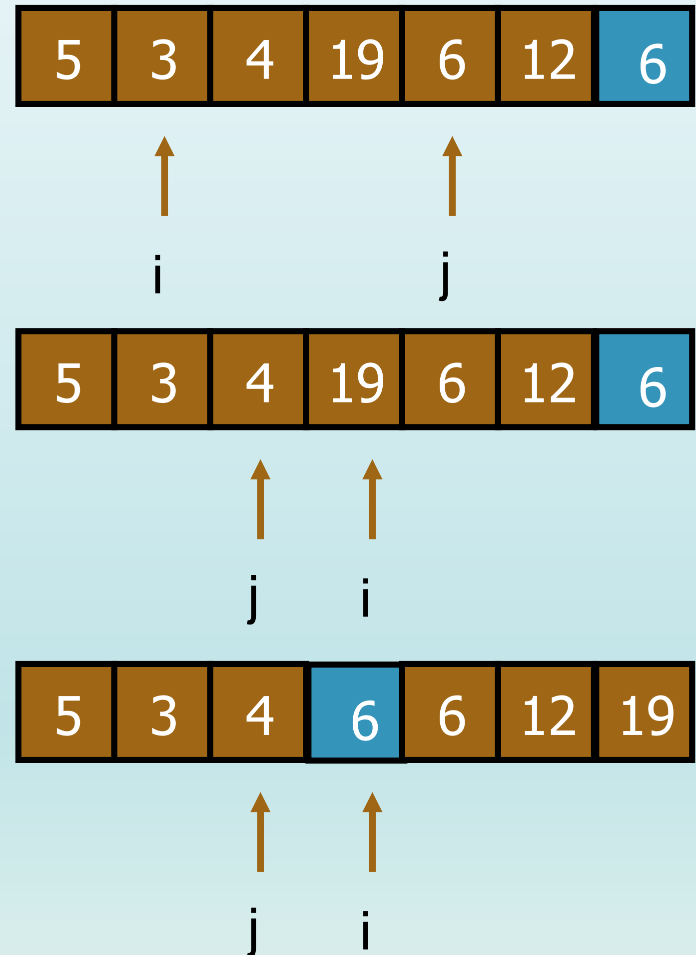
Partitioning Strategy

- When i and j have stopped and i is to the left of j
  - Swap A[i] and A[j]
  - After swapping
    - A[i] <= pivot
    - A[j] >= pivot
  - Repeat the process until i and j cross



Partitioning Strategy

- When i and j have crossed
  - Swap A[i] and pivot
- Result
  - A[p] <= pivot, for p < i
  - A[p] >= pivot, for p > i



Small arrays

- Cutoff value for small arrays
- Depends on
  - Time to make a recursive call
  - Architecture
  - Compiler
  - Other factors

Picking the Pivot

- Use the first element as pivot
  - OK if the input is random
  - What happens if the input is already sorted?
  - Can result in $O(N^2)$ behavior
- Choose the pivot randomly
  - Generally safe
  - Random number generation can be expensive

Picking the Pivot

- Use the median of the array
  - Partitioning always cuts the array into roughly half
  - An *optimal* quicksort (O(N log N))
  - How do you find the median?

Pivot: Median of Three

- Compare just three elements: the leftmost, rightmost and center
- Swap these elements if necessary so that
  - A[left] = Smallest
  - A[right] = Largest
  - A[center] = Median of three
- Pick A[center] as the pivot
- Swap A[center] and A[right – 1] so that pivot is at second-to-last position

median3

```
int center = ( left + right ) / 2;
if( a[ center ] < a[ left ] )
    swap( a[ left ], a[ center ] );
if( a[ right ] < a[ left ] )
    swap( a[ left ], a[ right ] );
if( a[ right ] < a[ center ] )
    swap( a[ center ], a[ right ] );

    // Place pivot at position right - 1
swap( a[ center ], a[ right - 1 ] );
```

Pivot: median of three

A[left] = 2, A[center] = 13,
A[right] = 6

Swap A[center] and A[right]

Choose A[center] as pivot

pivot

Swap pivot and A[right – 1]

pivot

Only need to partition A[left + 1, …, right – 2]. Why?

Main Quicksort Routine

```
if( left + 10 <= right )
{
    Comparable pivot = median3( a, left, right );

        // Begin partitioning
    int i = left, j = right - 1;
    for( ; ; )
    {
        while( a[ ++i ] < pivot ) { }
        while( pivot < a[ --j ] ) { }
        if( i < j )
            swap( a[ i ], a[ j ] );
        else
            break;
    }

    swap( a[ i ], a[ right - 1 ] );   // Restore pivot

    quicksort( a, left, i - 1 );      // Sort small elements
    quicksort( a, i + 1, right );     // Sort large elements
}
else   // Do an insertion sort on the subarray
    insertionSort( a, left, right );
```

Choose pivot

Partitioning

Recursion

For small arrays

Partitioning Part

- Works only if pivot is picked as *median-of-three*
  - A[left] <= pivot and A[right] >= pivot
  - Thus, only need to partition A[left + 1, …, right – 2]
- j will not run past the end
  - because A[left] <= pivot
- i will not run past the end
  - because A[right-1] = pivot

```
int i = left, j = right - 1;
for( ; ; )
{
    while( a[ ++i ] < pivot ) { }
    while( pivot < a[ --j ] ) { }
    if( i < j )
        swap( a[ i ], a[ j ] );
    else
        break;
}
```

Quicksort *vs.* Mergesort

- quicksort and mergesort are O(N log N) in the average case
- Why is quicksort *faster* than mergesort?
  - Inner loop
  - No extra juggling as in mergesort

```
int i = left, j = right - 1;
for( ; ; )
{
    while( a[ ++i ] < pivot ) { }
    while( pivot < a[ --j ] ) { }
    if( i < j )
        swap( a[ i ], a[ j ] );
    else
        break;
}
```

inner loop

Analysis

- Assumptions
  - A random pivot (no median-of-three partitioning)
  - No cutoff for small arrays

- Running time
  - pivot selection: constant time $O(1)$
  - partitioning: linear time $O(N)$
  - running time of two recursive calls

- $T(N) = T(i) + T(N – i - 1) + cN$ where c is a constant
  - i: number of elements in S1

Worst-Case Analysis

- What will be the worst case?
  - Pivot is smallest element, all the time
  - Partition is always unbalanced

$$
\begin{aligned}
T(N) &= T(N-1) + cN \\
T(N-1) &= T(N-2) + c(N-1) \\
T(N-2) &= T(N-3) + c(N-2) \\
&\vdots \\
T(2) &= T(1) + c(2) \\
T(N) &= T(1) + c\sum_{i=2}^{N} i = O(N^2)
\end{aligned}
$$

Best-case Analysis

- What will be the best case?
  - Partition is perfectly balanced

  - Pivot is always in the middle (median of the array)

$$\begin{aligned}
T(N) &= 2T(N/2) + cN \\
\frac{T(N)}{N} &= \frac{T(N/2)}{N/2} + c \\
\frac{T(N/2)}{N/2} &= \frac{T(N/4)}{N/4} + c \\
\frac{T(N/4)}{N/4} &= \frac{T(N/8)}{N/8} + c \\
&\vdots \\
\frac{T(2)}{2} &= \frac{T(1)}{1} + c \\
\frac{T(N)}{N} &= \frac{T(1)}{1} + c \log N \\
T(N) &= cN \log N + N = O(N \log N)
\end{aligned}$$

Efficiency of Quick Sort

- Best and worst case recurrences
  - Another way to solve them

- Best
  - $T(N) = 2 * T(N/2) + N; T(1) = 1$
  - $T(N) = 2 * (2 * T(N/4) + N/2) + N$

    $= 2 * (2 * (2 * T(N/8) + N/4) + N/2) + N$

    $= 2^k * T(N/2^k) + k * N$
  - $k = \lg(N) \rightarrow T(N) = N * T(1) + \lg(N) * N$

    $= O(N \lg (N))$
- Worst
  - $T(N) = T(N-1) + N; T(1) = 1$
  - $T(N) = ?$

Average-Case Analysis

- Difficult

- Average running time is $O(N \lg N)$

Finding k^{th} Smallest Element

- Compute k^{th} order statistic
  - k = 1 is min
  - k = N is max

- Partition based on some pivot value
  - i = partition (A, left, right)
  - if i matches k return A[i]   // Assuming 1-based
  - Recurse on left if k < i
  - Recurse on right if k > i
    - Find $(k – i)^{th}$ smallest