CSCI 340: Computational Models

Regular Languages
Regular Languages cool

If we can define a language by RE, then it’s a *regular language*

**Theorem**

*If* $L_1$ *and* $L_2$ *are regular languages, then* $L_1 + L_2$ *(union)*, $L_1L_2$ *(concatenation)*, and $L_1^*$ *(closure)* *are also regular languages.*

**Proof by Regular Expression.**

1. There exists REs $r_1$ and $r_2$ that define the regular languages $L_1$ and $L_2$
2. There exists an RE $(r_1 + r_2)$ that defines the language $L_1 + L_2$
3. There exists an RE $r_1r_2$ that defines the language $L_1L_2$
4. There exists an RE $r_1^*$ that defines the language $L_1^*$
5. All three of these sets of words are definable by RE

The set of regular languages is *closed* under union, concatenation, and Kleene closure.
Let us assume $TG_1$ and $TG_2$ exist that define languages $L_1$ and $L_2$ where each TG has a unique start and final state.

2. $L_1 + L_2$ can be described by:

3. $L_1L_2$ can be described by:

4. $L_1^*$ can be described by:
Proof by Machines

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Small problem for $L_1^*$ when the start has incoming edges. We must replicate the start state. We could convert to $\text{FA-}\lambda$ then to FA.
Example

\[ \Sigma = \{a, b\} \]

\[ L_1 = \text{all words of 2+ letters that begin and end with the same letter} \]

\[ L_2 = \text{all words that contain the substring} \ aba \]

\[ r_1 = a(a + b)^*a + b(a + b)^*b \]

\[ r_2 = (a + b)^*aba(a + b)^* \]

\[ r_1 + r_2 = \]

\[ r_1r_2 = \]

\[ r_1^* = \]
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\[r_1 + r_2 = [a(a + b)^*a + b(a + b)^*b] + [(a + b)^*aba(a + b)^*]\]

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$r_1^* = [a(a+b)^*a + b(a+b)^*b]^*$
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\[ r_1^* = [a(a + b)^*a + b(a + b)^*b]^* \]

Show the TGs that accept \( L_1 \) and \( L_2 \)
Show \( TG_1 + TG_2, TG_1 TG_2, \) and \( TG_1^* \)
Complements and Intersections

**Definition**

If $L$ is a language over alphabet $\Sigma$, we define its complement, $L'$ to be the language of all strings of letters from $\Sigma$ that are not words in $L$.

**Example**

If $L$ is the language over the alphabet $\Sigma = \{a, b\}$ of all words that have a double $a$ in them, then $L'$ is the language of all words that do not have a double $a$.

We must specify the alphabet $\Sigma$ or else the complement of $L$ might contain $cat$, $dog$, $\ldots$ (because they are definitely not strings in $L$).

$$(L')' = L$$

for obvious reasons (theorem in set theory)
Complements and Regular Languages

**Theorem**

If \( L \) is a regular language, then \( L' \) is also a regular language. In other words, the set of regular languages is closed under complementation.

**Proof.**

- If \( L \) is a regular language, we know from Kleene’s theorem that there is some FA that accepts \( L \).
- The states of FA are each either final or non-final.
- Let us reverse the final status of each state (e.g. final \( \rightarrow \) non-final, non-final \( \rightarrow \) final).
- This new machine accepts all input strings the original FA rejected (\( L' \)). Likewise, the new machine rejects all input strings the original FA accepted (\( L \)).
- This new FA can be converted to an RE via Kleene’s theorem □
Complements of Regular Languages Example
Complements of Regular Languages Example
Language Intersection

**Theorem**

If $L_1$ and $L_2$ are regular languages, than $L_1 \cup L_2$ is also a regular language. e.g. the set of regular languages is closed under intersection.
Language Intersection

From the above, it is obvious how \((L'_1 + L'_2)' = L_1 \cup L_2\)
Algorithm for finding RE accepting $L_1 + L_2$

**Algorithm**

1. Define $r_1$ and $r_2$ which represent $L_1$ and $L_2$
2. Convert $r_1$ and $r_2$ to $FA_1$ and $FA_2$
3. Invert the states of $FA_1$ and $FA_2$ resulting in $FA'_1$ and $FA'_2$
4. Merge $FA'_1$ and $FA'_2$ into $TG'$, then convert $TG'$ into $FA'_3$
5. Invert the states of $FA'_3$, resulting in $FA_3$ (which accepts $L_1 \cup L_2$)

**Proof.**
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Proof.

1. For a regular language, there exists a RE
2. Given an RE, there exists an FA (Kleene’s theorem)
3. We can complement an FA by swapping its states
4. We can describe $L'_1 + L'_2$ by merging two TGs
5. We can convert a TG to an RE
Example

$L_1 = \text{all strings with a double } a$

$L_2 = \text{all strings with an even number of } a\text{'s}$
Example

$L_1 = \text{all strings with a double } a$

$L_2 = \text{all strings with an even number of } a \text{'s}$

We can define $L_1$ and $L_2$ by the following REs:

\[
\begin{align*}
    r_1 &= (a + b)^* aa (a + b)^* \\
    r_2 &= b^* (ab^* ab^*)^* 
\end{align*}
\]
Example

$L_1 = \text{all strings with a double } a$

$L_2 = \text{all strings with an even number of } a\text{’s}$

We can define $L_1$ and $L_2$ by the following REs:

\[ r_1 = (a + b)^* aa (a + b)^* \]

\[ r_2 = b^* (ab^* ab^*)^* \]

Or the following FAs:
Example

Swapping the states:

Merging (Creating the TG):

[Diagram showing the state transition graphs after swapping and merging]
Example

After converting the TG to FA:
Example

After swapping all of the states:

And converting the FA to RE with the bypass algorithm:

\[(a + ab^*ab)^*a(a + bb^*aab^*a)(a + ab^*a)^*\]
A Better Way...

- Remember creating a machine that accepts $FA_1 + FA_2$ where $FA_1$ has $x$-states, $FA_2$ has $y$-states, and our new machine has $z$-states
- We identify all final $z$-states by $x$-or-$y$ states being accepted upon the construction of our new machine
- Let’s change the designation for $FA_1 \cup FA_2$ to: All final $z$-states by $x$-and-$y$ states being accepted upon the construction of our new machine
- Now the new FA accepts only strings that reach simultaneously on both machines

**TL;DR** – change the rules of determining a final state of two FAs to be the intersection ($\cup$) rather than union ($+$)
One Final Example

Our two languages will be:

\[ L_1 = \text{all words that begin with an } a \]
\[ L_2 = \text{all words that end with an } a \]
\[ r_1 = a(a + b)^* \]
\[ r_2 = (a + b)^*a \]

An obvious solution is:

\[ a(a + b)^*a + a \]

But now we need to prove it...
Homework 6a

For each of the following pairs of regular languages, find a RE and FA that define $L_1 \cup L_2$

1. $(a + b)^*a \quad b(a + b)^*$

2. Even-length strings $(b + ab)^*(a + \lambda)$

3. Odd-length strings $a(a + b)^*$

4. Even-length strings Strings with an even number of $a$’s